Compatible Poisson-Lie structures on the loop group of SL_2.

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Abstract. We define a 1-parameter family of r-matrices on the loop algebra of sl_2, defining compatible Poisson structures on the associated loop group, which degenerate into the rational and trigonometric structures, and study the Manin triples associated to them.

Introduction. The concept of bi-(or multi-)Hamiltonian structures on manifolds is known, since the works of Magri and Gelfand-Dorfman ([Ma], [GD]), to play an important role in the study of classical integrable systems. Up to now, it still remains unclear whether this notion has any reasonable quantum analogue. Among long-known examples of such structures are the KdV hierarchy phase spaces. In that case, higher compatible Poisson structures are nonlocal, and are expected to be quantized using (nonlocal) vertex operator algebras. This program is very far from its goals at the moment.

An other example of bi-Hamiltonian manifolds comes from the theory of Poisson-Lie groups. It is an elementary fact that the rational and trigonometric structures (according to the terminology of [BD]) on loop groups, are compatible. Accordingly, it is a natural question to find higher compatible Poisson structures on these groups. In this paper, we answer this question in the sl_2 case. We perform an elementary manipulation on the rational r-matrix, that after the operation of shifting of the spectral parameter generates the rational and trigonometric r-matrices. This shows the compatibility of these Poisson structures. We then study the associated Manin triples. They are of a slightly different kind from those studied in [D]. It would be interesting to achieve explicitly the quantization of the structures found here.

A possible connection between the two problems mentioned above could be the remark, that the Poisson brackets of the KdV monodromy operators in the first (resp. second) Hamiltonian structures are respectively given by the rational and trigonometric r-matrices ([FT]).

1. Compatible r-matrices for $sl_2(\mathbf{C}((\lambda^{-1})))$.

Let e, f, h be the Chevalley basis of $a = sl_2(\mathbf{C})$, $t = h \otimes h + 2(e \otimes f + f \otimes e)$ the split Casimir element of $sl_2(\mathbf{C})$. The rational and trigonometric r-matrices of $a(\mathbf{C}((\lambda)))$ are

$$r_1(\lambda, \mu) = \frac{t}{\lambda - \mu} \quad \text{and} \quad r_2(\lambda, \mu) = \frac{\lambda + \mu}{\lambda - \mu} t + 2(e \otimes f + f \otimes e)$$

They can be considered as elements of $Usl_2(\mathbf{C})^{\otimes 2} \otimes \mathbf{C}(\lambda, \mu)$, and satisfy the Yang-Baxter equation with parameters, that is

$$[\![r, r]\!](\lambda, \mu, \nu) := [r^{12}(\lambda, \mu), r^{13}(\lambda, \nu)] + \text{cyclic permutations} = 0,$$

in the algebra $Usl_2(\mathbf{C})^{\otimes 3} \otimes \mathbf{C}(\lambda, \mu, \nu)$. Let us perform the following transformation on r_1: we replace λ and μ by their inverses and apply the affine Weyl group transformation $\lambda^k e \to \lambda^{k+1} e$, $\lambda^k h \to \lambda^k h$, $\lambda^k f \to \lambda^{k-1} f$. Up to sign the resulting r-matrix is

$$r_3(\lambda, \mu) = \frac{\lambda\mu}{\lambda - \mu} h \otimes h + 2\frac{\lambda^2}{\lambda - \mu} e \otimes f + 2\frac{\mu^2}{\lambda - \mu} f \otimes e = \frac{\lambda\mu}{\lambda - \mu} t + 2\lambda e \otimes f - 2\mu f \otimes e \quad (1)$$

Adding a quantity E to λ and μ transforms r_3 into $r_3 + Er_2 + E^2 r_1$, so $[\![r_3 + Er_2 + E^2 r_1, r_3 + Er_2 + E^2 r_1]\!] = 0$. From this follows that $[\![r_1, r_2]\!] = [\![r_2, r_3]\!] = 0$ and also that $[\![r_1, r_3]\!] = 0$ (since we know that $[\![r_2, r_2]\!] = 0$; we set $[\![a, b]\!] = [a^{12}(\lambda, \mu), b^{13}(\lambda, \nu)] + [b^{12}(\lambda, \mu), a^{13}(\lambda, \nu)]$+c.p.). Hence, the Poisson structures defined on $SL_2(\mathbf{C}((\lambda)))$ by r_1, r_2 and r_3 are compatible (in the sense of [Ma], [GD]). Let $\tau_E^{(0)}$ be the automorphism of $\mathbf{C}((\lambda^{-1}))$ defined by $\tau_E(\lambda^k) = (\lambda + E)^k = \lambda^k + kE\lambda^{k-1} + ...$ and let us call τ_E the automorphism induced on $SL_2(\mathbf{C}((\lambda^{-1})))$. We have:

Proposition. *For a_i complex numbers ($i = 1, 2, 3$), the multiplicative bivectors defined by $\sum_{i=1}^3 a_i r_i$ define structures of Poisson-Lie group on $SL_2(\mathbf{C}((\lambda^{-1})))$. Up to multiplication and the action of the automorphisms τ_E, this family is reduced to a 1-parameter family, with special points the rational and trigonometric structures.*

2. Modified Yang-Baxter equation.

Let us remark that solutions to the Yang-Baxter equation with parameters usually correspond to solutions of the modified Yang-Baxter equation, after interpretation of r-matrices as elements of $[sl_2(\mathbf{C}((\lambda^{-1}))) \otimes sl_2(\mathbf{C}((\mu^{-1})))]^-$ ($^-$ means direct product of homogeneous components, where the grading is $\deg(x\lambda^k \otimes 1) = \deg(1 \otimes x\mu^k) = k$ if $x \in sl_2(\mathbf{C})$ and $k \in \mathbf{Z}$, and $\deg \lambda = \deg \mu = 1$). Indeed, by this interpretation the r-matrices r_i, $i = 1, 2, 3$ correspond respectively to

$$\bar{r}_1 = \frac{1}{2}(\frac{1}{\lambda - \mu} - \frac{1}{\mu - \lambda})t, \bar{r}_2 = \frac{1}{2}(\frac{\lambda + \mu}{\lambda - \mu} - \frac{\lambda + \mu}{\mu - \lambda})t + 2(e \otimes f - f \otimes e),$$

$$\bar{r}_3 = \frac{1}{2}(\frac{\lambda\mu}{\lambda - \mu} - \frac{\lambda\mu}{\mu - \lambda})t + 2\lambda e \otimes f - 2\mu f \otimes e \quad (2)$$

(we use the convention $\frac{1}{\lambda - \mu} = \sum_{i \geq 0} \mu^i / \lambda^{i+1}$).

Let us now consider the tensors

$$t_1 = \frac{1}{2}(\frac{1}{\lambda - \mu} + \frac{1}{\mu - \lambda})t, t_2 = \frac{1}{2}(\frac{\lambda + \mu}{\lambda - \mu} + \frac{\lambda + \mu}{\mu - \lambda})t, t_3 = \frac{1}{2}(\frac{\lambda\mu}{\lambda - \mu} + \frac{\lambda\mu}{\mu - \lambda})t$$

They can be considered as the split Casimir elements corresponding to the invariant pairings on $sl_2(\mathbf{C}((\lambda^{-1})))$

$$\langle a(\lambda), b(\lambda) \rangle_1 = \mathrm{res}_{\lambda=0} \langle a(\lambda), b(\lambda) \rangle d\lambda, \langle a(\lambda), b(\lambda) \rangle_2 = \frac{1}{2} \mathrm{res}_{\lambda=0} \langle a(\lambda), b(\lambda) \rangle \frac{d\lambda}{\lambda},$$

$$\langle a(\lambda), b(\lambda) \rangle_3 = \mathrm{res}_{\lambda=0} \langle a(\lambda), b(\lambda) \rangle \frac{d\lambda}{\lambda^2},$$

which allow the interpretation of $(sl_2(\mathbf{C}((\lambda^{-1}))), \bar{r}_i)$ as double algebras (see [D]); here $\langle , \rangle$ is the Killing form on $sl_2(\mathbf{C})$.

Lemma. *Let a_i be numbers ($i = 1, 2, 3$), then we have*

$$[\![\sum_i a_i \bar{r}_i, \sum_i a_i \bar{r}_i]\!] = [\![\sum_i a_i t_i, \sum_i a_i t_i]\!]. \tag{3}$$

Proof. We first show (3) for $a_1 = a_2 = 0$: in this case, it follows from the structure of the Manin triple in the rational case; we can also see it as a consequence of the identity (in formal series) $\frac{\lambda\mu}{\lambda-\mu}\frac{\mu\nu}{\mu-\nu} + \text{p.c.} = 0$ (itself a consequence of $\frac{1}{\mu^{-1}-\lambda^{-1}}\frac{1}{\nu^{-1}-\mu^{-1}} + \text{p.c.} = 0$). We prove (3) for $a_1 = a_3 = 0$ and $a_2 = a_3 = 0$ similarly. Returning now to the identity for $a_2 = a_3 = 0$, the action of τ_E transforms t_3 into $t_3 + Et_2 + E^2 t_1$, and $\bar{r}_3$ into $\bar{r}_3 + E\bar{r}_2 + E^2\bar{r}_1$, so we have $[\![\bar{r}_3 + E\bar{r}_2 + E^2\bar{r}_1, \bar{r}_3 + E\bar{r}_2 + E^2\bar{r}_1]\!] = [\![t_3 + Et_2 + E^2 t_1, t_3 + Et_2 + E^2 t_1]\!]$; so we deduce from $[\![\bar{r}_i, \bar{r}_j]\!] = [\![t_i, t_j]\!]$ for $i, j \le 2$ or $i, j \ge 2$, the same identity for all i, j. $\square$

4. Manin triples.

Let us now consider $g = sl_2(\mathbf{C}((\lambda^{-1})))$, endowed with the Poisson-Lie structure given by $\sum a_i r_i$. Our aim in this section is to describe it as a double algebra. For this, and following [STS], we study the eigenvalues of the operator $R : g \to g$ defined by $RA = \langle \sum_i a_i r_i, A \otimes 1\rangle_{(a_i)}$, where $\langle,\rangle_{(a_i)}$ is the scalar product corresponding to $\sum a_i t_i$

Let us compute $\langle,\rangle_{(a_i)}$. $\sum_i a_i t_i = \sum a'_{i+j+2}(2e\lambda^i \otimes f\mu^j + 2f\lambda^i \otimes e\lambda^j + h\lambda^i \otimes h\mu^j)$, with $a'_{1,3} = a_{1,3}$ and $a'_2 = 2a_2$. The inverse of the matrix A with coefficients $A_{ij} = a_{i+j+2}$ has coefficients $(A^{-1})_{ij} = b_{i+j}$, where b_n is the coefficient of λ^{-n} in the expansion of $(a_1\lambda^{-1} + a_2 + a_3\lambda)^{-1}$. We will have $\langle e\lambda^i, f\mu^j\rangle_{(a_i)} = \frac{1}{2}\langle h\lambda^i, h\mu^j\rangle_{(a_i)} = b_{i+j}$, and so

$$\langle A(\lambda), B(\lambda)\rangle_{(a_i)} = \text{res}_{\lambda=\infty}\langle A(\lambda), B(\lambda)\rangle \frac{d\lambda}{a_1 + 2a_2\lambda + a_3\lambda^2} \tag{4}$$

We can then express R as

$$\begin{aligned} R(A)(\mu) &= \langle r(\lambda,\mu), A(\lambda)\rangle_{(a_i)} \\ &= \text{res}_{\lambda=\infty}\{t(a_1 + (\lambda+\mu)a_2 + \lambda\mu a_3)\frac{1}{2}(\frac{1}{\lambda-\mu} - \frac{1}{\mu-\lambda}) + 2(a_2 + a_3\lambda)e \otimes f \\ &- 2(a_2 + a_3\mu)f \otimes e\}A(\lambda)\frac{d\lambda}{a_1 + 2a_2\lambda + a_3\lambda^2} \\ &= \text{res}_{\lambda=\infty}\, tA(\lambda)\frac{1}{2}(\frac{1}{\lambda-\mu} - \frac{1}{\mu-\lambda})d\lambda + \text{res}_{\lambda=\infty}\frac{-t(a_2 + a_3\lambda)}{a_1 + 2a_2\lambda + a_3\lambda^2}A(\lambda)d\lambda \\ &+ \text{res}_{\lambda=\infty}\frac{2e \otimes f(a_2 + a_3\lambda)}{a_1 + 2a_2\lambda + a_3\lambda^2}A(\lambda)d\lambda - \text{res}_{\lambda=\infty}\frac{2f \otimes e(a_2 + a_3\mu)}{a_1 + 2a_2\lambda + a_3\lambda^2}A(\lambda)d\lambda, \end{aligned}$$

where $\langle,\rangle$ is understood in the two last lines. Since $\langle t, A\rangle = 2A$ ($\langle a, b\rangle$ is the trace of ab in the fundamental representation), and denoting for a formal series $\phi = \sum \phi_i\lambda^i$, $\phi_{>0} = \sum_{i>0}\phi_i\lambda^i$ and $\phi_{\le 0} = \sum_{i\le 0}\phi_i\lambda^i$, we get for the first term $A(\mu)_{>0} - A(\mu)_{\le 0}$.

We see that R acts as 1 on $a \otimes \mathbf{C}[[\lambda^{-1}]]$, and -1 on $f \otimes \lambda\mathbf{C}[\lambda]$. In addition, the subspace $h \otimes \mathbf{C}[\lambda] \oplus e \otimes \lambda^{-1}\mathbf{C}[\lambda]$ is stable under R. Let $A = \sum_{i\ge 0}\alpha_i h\lambda^i + \sum_{i\ge -1}\beta_i e\lambda^i$ belong to this space. We now restrict ourselves to the particular case $a_1 = 1$, $a_2 = 0$ and $a_3 = -1$. Pose $RA = \sum_{i\le 0}\alpha'_i h\lambda^i + \sum_{i\le 1}\beta'_i e\lambda^i$. Then

$$\alpha'_0 = \alpha_0 + 2(\alpha_2 + \alpha_4 + \cdots), \beta'_0 = \beta_0 + 2(\beta_2 + \beta_4 + \cdots), \beta'_1 = \beta_{-1} + 2(\beta_1 + \beta_3 + \cdots),$$

and

$$\alpha_i' = -\alpha_i, \quad \beta_i' = -\beta_i \text{ for } i < 0.$$

The space is then the sum of the eigenspaces $\mathbf{C}h \oplus \mathbf{C}e \oplus \mathbf{C}e\lambda^{-1}$ for the eigenvalue 1, and $\{A(\lambda)|\alpha_0 + \alpha_2 + \cdots = 0, \beta_0 + \beta_2 + \cdots = 0, \beta_{-1} + \beta_1 + \beta_3 \cdots = 0\}$ for -1. So we have:

Proposition. *The eigenspaces of R are $a[[\lambda^{-1}]]$ and g_+, where g_+ is the subspace of $a \otimes \mathbf{C}(\lambda)$ of functions $A(\lambda)$ with only poles zero and infinity, and such that $A(1)$ and $A(-1) \in b_-$, $[A(1) + A(-1)]_h = 0$, and $A \in \frac{1}{\lambda} n_+ \oplus b_+ \oplus \lambda a[[\lambda]]$ (in the completion at 0 $a \otimes \mathbf{C}((\lambda))$). Here $n_+ = \mathbf{C}e$, $\underline{h} = \mathbf{C}h$, $b_+ = \underline{h} \oplus n_+$, $b_- = \mathbf{C}h + \mathbf{C}f$, $[\ ]_h$ is the natural projection from b_- to $\underline{h}$.*

These two spaces are supplementary Lie subalgebras of g, isotropic for the form

$$\langle A(\lambda), B(\lambda)\rangle_{(1,0,-1)} = \mathrm{res}_{\lambda=\infty}\langle A(\lambda), B(\lambda)\rangle \frac{d\lambda}{1-\lambda^2}.$$

So the decomposition $g = g_+ \oplus a[\lambda]$, together with the above scalar product on g, defines a Manin triple.

These last facts can be checked directly, for example for the isotropy of g_+, the residue formula gives: if A and $B \in g_+$, then

$$\begin{aligned}
\langle A(\lambda), B(\lambda)\rangle_{(a_i)} &= -\mathrm{res}_{\lambda=1}\langle A, B\rangle \frac{d\lambda}{\lambda-1}\frac{-1}{\lambda+1} - \mathrm{res}_{\lambda=-1}\langle A, B\rangle \frac{d\lambda}{\lambda+1}\frac{-1}{\lambda-1} \\
&\quad - \mathrm{res}_{\lambda=0}\langle A, B\rangle \frac{d\lambda}{1-\lambda^2} \\
&= \frac{1}{2}(\mathrm{res}_{\lambda=1}\langle A, B\rangle \frac{d\lambda}{\lambda-1} - \mathrm{res}_{\lambda=-1}\langle A, B\rangle \frac{d\lambda}{\lambda+1}) \\
&\quad - \mathrm{res}_{\lambda=0}\langle \frac{1}{\lambda} n_+^A + b_+^A + \cdots, \frac{1}{\lambda} n_+^B + b_+^B + \cdots \rangle d\lambda (1 + \lambda^2 + \cdots)
\end{aligned}$$

The first term is zero by the compatibility conditions at points 1 and -1, and the second vanishes also.

Remarks.

1. Remind, that a solution $r(\lambda, \mu)$ of CYBE is called non-degenerate if it is non-degenerate for at least one point (λ, μ). Drinfeld conjectered that every non-degenerate rational r-matrix $r(\lambda, \mu)$ is equivalent to a solution of CYBE of the form

$$r(\lambda, \mu) = \frac{t}{\lambda - \mu} + P(\lambda, \mu),$$

where $P(\lambda, \mu)$ is a polynomial of degree at most one in each variables. This problem was investigated by A. Stolin in [S] and it follows from his results that in the $sl_2(\mathbf{C})$-case there are two rational r-matrices:

$$r_1'(\lambda, \mu) = \frac{t}{\lambda - \mu} + 2(h \otimes e - e \otimes h), \tag{5}$$

$$r_1''(\lambda,\mu) = \frac{t}{\lambda-\mu} + 2(\mu h\otimes e - \lambda e\otimes h). \tag{6}$$

We can perform the manipulations of sect. 1 on the above matrices. Starting with the r-matrices (5) and (6), we obtain the r-matrices

$$r_3^{'}(\lambda,\mu) = \frac{\lambda\mu t}{\lambda-\mu} + 2(\lambda e\otimes f + \mu f\otimes e) + 2(\mu h\otimes e - \lambda e\otimes h)$$

and

$$r_3^{''}(\lambda,\mu) = \frac{\lambda\mu t}{\lambda-\mu} + 2(\lambda e\otimes f + \mu f\otimes e) + 2(h\otimes e - e\otimes h),$$

which can be included in the similar compatible triple of multiplicative Poisson structures (the bialgebra structure corresponding to r_1' has been quantized recently in [KhST]).

2. In the case of different parameters (a_i), the points 1 and -1 should be shifted to some other points z_i of $\mathbf{C}^*$ (in the same time the form $\frac{d\lambda}{\lambda^2-1}$ is changed to $\frac{d\lambda}{(\lambda-z_1)(\lambda-z_2)}$). It would be interesting to see how this family a Manin triples degenerates to the known cases rational and trigonometric ones (as they are described in [D]).

3. It could also be interesting to generalize this to a situation with more marked points. We see that the description given here does not fit in the adelic framework if [D] (the form is not the sum of residues at all singular points but only the residue at one point, and the "evident" half of Manin pair is not the algebra of functions regular outside these points).

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References

[BD] A.A. Belavin, V.G. Drinfeld, *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, Funct. An. Appl. **16:3** (1981), 159–80.

[D] V.G. Drinfeld, *Quantum groups*, Proc. ICM-86 **1** (1987), 798–820.

[FT] L.D. Faddeev, L.A. Takhtajan, *Hamiltonian methods in soliton equations*, Springer-Verlag, Berlin, 1992.

[GD] I.M. Gelfand, I.Ya. Dorfman, *Schouten brackets and Hamiltonian operators*, Funct. An. Appl. **14:3** (1980), 71–4.

[KhST] S.M. Khoroshkin, A.A. Stolin, V.N. Tolstoy, *Deformation of Yangian $Y(sl_2)$*, q-alg/9511005 (1995).

[M] F. Magri, *A geometric approach to the nonlinear solvable equations*, Lect. Notes in Physics **120** (1980), Springer, 233–63.

[STS] M.A. Semenov-Tian-Shansky, *What is a classical r-matrix?*, Funct. An. Appl. **17** (1983), 259–72.

[S] A.Stolin, *On rational solutions of Yang-Baxter equations. Maximal orders in loop algebras*, Commun. Math. Phys. **141(3)** (1991), 533–48.

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